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Asymptotic operator algebras in quantum mechanics

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Abstract. The asymptotic behaviour of quantum mechanical states at large times has recently been discussed by Wan and McLean. This paper deals with the corresponding behaviour of quantum mechanical operators. The concept of asymptotic operators is introduced and their mathematical properties in the weak, the uniform and the strong operator topologies studied. Results are presented in a series of theorems, lemmas and corollaries.

1. Introduction

It is well known (Amrein 1981) that every state of a free particle is a scattering state, i.e. the probability of finding a free particle in any fixed bounded region in the configuration space $\mathbb{R}^n$ vanishes as time t tends to infinity.

We have made a study of the behaviour of quantum mechanical states at large times (Wan and McLean 1983a, b) and in this paper we shall present some mathematical properties of the asymptotic behaviour of observables. This is not a purely mathematical exercise; in fact we shall, in an ensuing paper, propose a formulation of quantum mechanics based on one of the operator algebras introduced here.

We confine our attention to a free quantum system of mass m having configuration space $\mathbb{R}^n$. A state ϕ is then a normalised member of the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$ and its time evolution is given by $\phi_t = U_t^0\phi$ where $U_t^0 = \exp(H_0 t/i\hbar)$ and H_0 is the n-dimensional free Hamiltonian.

We shall denote the von Neumann algebra of all bounded operators on $\mathscr{H}$ by $B(\mathscr{H})$.

For each A in $B(\mathscr{H})$ we denote $U_t^{0\dagger} A U_t^0$ by A_t. The spectral measures of position and momentum are denoted by E_x and E_p respectively.

Throughout this paper we shall on many occasions take limits as time tends to infinity so for brevity we write lim for $\lim_{t\to\infty}$.

2. The notion of asymptotic operator algebras

We shall give a general discussion here, leaving the study of particular asymptotic algebras to later sections.

Definition 1. Asymptotic operators and asymptotic operator algebras.

(1) An operator A in $B(\mathscr{H})$ will be called an asymptotic operator if $\lim\langle\phi|A_t\phi\rangle$ exists for every ϕ in $\mathscr{H}$.

(2) A C^*-subalgebra of $B(\mathscr{H})$ will be called an asymptotic operator algebra on $\mathscr{H}$ if every element of the algebra is an asymptotic operator.

These definitions can easily be extended to a simple scattering system with Hamiltonian H by replacing A_t by $U_t^\dagger A U_t$ where $U_t = \exp(Hti\hbar)$ and by requiring $\lim\langle\phi|U_t^\dagger A U_t\phi\rangle$ to exist for all scattering states ϕ. We shall only consider the case of a free Hamiltonian in this paper for simplicity.

The physical meaning of the above definitions is transparent. Literally they mean that the expectation value $\langle\phi_t|A\phi_t\rangle$ which equals $\langle\phi|A_t\phi\rangle$ converges in time. In other words we can talk about the expectation value of an asymptotic observable as the particle goes to 'infinity'. Examples of such operators are all spectral projectors $E_{\boldsymbol{p}}(\Lambda)$ of momentum $\boldsymbol{p}$ and also the spectral projectors $E_{\boldsymbol{x}}(\Lambda)$ of position $\boldsymbol{x}$ for a bounded Borel set Λ in $\mathbb{R}^n$.

The following theorem shows that not every operator in $B(\mathscr{H})$ is an asymptotic operator and consequently that an asymptotic algebra of operators is necessarily a proper subset of $B(\mathscr{H})$.

Theorem 1. Let $\psi \in \mathscr{H}$, $\psi \neq 0$; then there exists a spectral projector $E_{\boldsymbol{x}}(\Lambda)$ of position for which $\lim\langle\psi_t|E_{\boldsymbol{x}}(\Lambda)\psi_t\rangle$ does not exist. The sets Λ and $\mathbb{R}^n - \Lambda$ are necessarily unbounded.

Proof.

(i) Without loss of generality we may assume $\|\psi\| = 1$. Let b_r be a ball of radius r centred at the origin in $\mathbb{R}^n$. Let r_0 and t_0 be fixed and let $\varepsilon > 0$; then since ψ is a scattering state we have for some $T > t_0$

$$\|E_{\boldsymbol{x}}(b_{r_0})\psi_T\|^2 < \tfrac{1}{2}\varepsilon.$$

Also since $I - E_{\boldsymbol{x}}(b_r)$ converges strongly to 0 as $r \to \infty$ there exists $N > r_0$ such that

$$\|(I - E_{\boldsymbol{x}}(b_N))\psi_T\|^2 < \tfrac{1}{2}\varepsilon.$$

Now since $N > r_0$

$$\begin{aligned} 1 &= \|\{(E_{\boldsymbol{x}}(b_N) - E_{\boldsymbol{x}}(b_{r_0})) + [I - (E_{\boldsymbol{x}}(b_N) - E_{\boldsymbol{x}}(b_{r_0}))]\}\psi_T\|^2 \\ &= \|(E_{\boldsymbol{x}}(b_N) - E_{\boldsymbol{x}}(b_{r_0}))\psi_T\|^2 + [\|(I - E_{\boldsymbol{x}}(b_N))\psi_T\|^2 + \|E_{\boldsymbol{x}}(b_{r_0})\psi_T\|^2]. \end{aligned}$$

Hence given r_0, t_0 and $\varepsilon > 0$ there exist $N > r_0$ and $T > t_0$ such that

$$\|(E_{\boldsymbol{x}}(b_N) - E_{\boldsymbol{x}}(b_{r_0}))\psi_T\|^2 > 1 - \varepsilon.$$

(ii) Given $\varepsilon = r_0 = t_0 = 1$ we can use the result established in (i) to find $r_1 > r_0$ and $t_1 > t_0 + 1$ such that

$$\|(E_{\boldsymbol{x}}(\Lambda_{r_1}) - E_{\boldsymbol{x}}(\Lambda_{r_0}))\psi_{t_1}\|^2 > 1 - 1/1 = 0.$$

Applying the result again starting with r_1, $t_1 + 1$ and $\varepsilon = \frac{1}{2}$ we deduce that there exist $r_2 > r_1$ and $t_2 > t_1 + 1$ with

$$\|(E_{\boldsymbol{x}}(b_{r_2}) - E_{\boldsymbol{x}}(b_{r_1}))\psi_{t_2}\|^2 > 1 - \tfrac{1}{2}.$$

Repeating this process we obtain inductively an increasing sequence b_{r_k} converging to $\mathbb{R}^n$ and a sequence t_k diverging to $+\infty$ such that for each integer $k \geqslant 1$

$$\|(E_{\boldsymbol{x}}(b_{r_k}) - E_{\boldsymbol{x}}(b_{r_{k-1}}))\psi\|^2 > 1 - 1/k.$$

Let $\Delta_k = b_{r_k} - b_{r_{k-1}}$ and $\Delta = \Delta_2 \cup \Delta_4 \cup \ldots$; then $\mathbb{R}^n - \Delta = b_{r_0} \cup \Delta_1 \cup \Delta_3 \cup \ldots$. Now we have

$$\|E_x(\Delta)\psi_{t_{2k}}\|^2 \geq \|E_x(\Delta_{2k})\psi_{t_{2k}}\|^2 > 1 - 1/2k \geq \tfrac{1}{2},$$

$$\|E_x(\mathbb{R}^n - \Delta)\psi_{t_{2k+1}}\|^2 \geq \|E_x(\Delta_{2k+1})\psi_{t_{2k+1}}\|^2 > 1 - 1/(2k+1),$$

and it follows that

$$\|E_x(\Delta)\psi_{t_{2k+1}}\|^2 = 1 - \|E_x(\mathbb{R}^n - \Delta)\psi_{t_{2k+1}}\|^2 < 1/(2k+1).$$

Hence $\lim_{k\to\infty}\|E_x(\Delta)\psi_{t_{2k+1}}\|^2 = 0$ while $\|E_x(\Delta)\psi_{t_{2k}}\| \geq \frac{1}{2}$ for all k.

Hence $\lim_{t\to\infty}\langle\psi_t \mid E_x(\Delta)\psi_t\rangle$ does not exist since the above two subsequences do not converge to the same limit.

Finally it is easily checked from the definition of a scattering state that the limit exists whenever Δ or $\mathbb{R}^n - \Delta$ is bounded.

Definition 2. The set of asymptotic operators. We call the set

$$\mathscr{A}^w = \{A \in B(\mathscr{H}) \colon \lim\langle\phi \mid A_t\phi\rangle \text{ exists } \forall \phi \in \mathscr{H}\}$$

the set of asymptotic operators. We also introduce the set

$$\mathscr{A}_0^w = \{A \in B(\mathscr{H}) \colon \lim\langle\phi \mid A_t\phi\rangle = 0 \ \forall \phi \in \mathscr{H}\}.$$

The use of the superscript w is explained by the following theorem.

Theorem 2.

$$\mathscr{A}^w = \{A \in B(\mathscr{H}) \colon w\text{-lim } A_t \text{ exists}\},$$

$$\mathscr{A}_0^w = \{A \in B(\mathscr{H}) \colon w\text{-lim } A_t = 0\},$$

where w-lim denotes weak convergence.

Proof.

(i) An operator $A \in B(\mathscr{H})$ such that A_t converges weakly is obviously in $\mathscr{A}^w$.

(ii) From the identity

$$2\langle\phi \mid A\psi\rangle = \langle\phi + \psi \mid A(\phi + \psi)\rangle - \mathrm{i}\langle\phi + \mathrm{i}\psi \mid A(\phi + \mathrm{i}\psi)\rangle + \mathrm{i}\langle\phi \mid A\phi\rangle$$
$$- \mathrm{i}\langle\psi \mid A\psi\rangle - \langle\phi \mid A\phi\rangle - \langle\psi \mid A\psi\rangle$$

which holds for any A in $B(\mathscr{H})$ and any ϕ, ψ in $\mathscr{H}$ we deduce that $\lim\langle\phi \mid A_t\psi\rangle$ exists for every A in $\mathscr{A}^w$. For a fixed A the formula $s(\phi, \psi) = \lim\langle\phi \mid A_t\psi\rangle$ defines a sesquilinear form s on $\mathscr{H}$ which is bounded since

$$|s(\phi, \psi)| = \lim|\langle\phi \mid A_t\psi\rangle| \leq \|A\|\|\phi\|\|\psi\|.$$

It follows (Weidmann 1980) that there is an operator $T \in B(\mathscr{H})$ with $s(\phi, \psi) = \langle\phi \mid T\psi\rangle$ for all $\phi, \psi \in \mathscr{H}$ and clearly A_t converges weakly to T.

(iii) The result for $\mathscr{A}_0^w$ follows directly from the identity above.

Lemma 1. The Fourier–Plancherel operator F satisfies

$$\lim\langle\phi \mid F_t\psi\rangle = 0 \qquad \forall \phi, \psi \in \mathscr{H}.$$

Proof. The Fourier–Plancherel operator F is a unitary operator on $\mathscr{H} = L^2(\mathbb{R}^n)$ (Roman

1975), and it relates the position and momentum operators by $\boldsymbol{p} = \hbar F^{\dagger}\boldsymbol{x}F$ (Prugovečki 1971).

Let $V_t = \exp(\hbar t\boldsymbol{x}^2/\mathrm{i}2m)$; then $U_t^0 = F^{\dagger}V_tF$. Now let us assume, without loss of generality, that $\|\phi\| = \|\psi\| = 1$. Then for any $\varepsilon > 0$ we can choose a bounded Borel set Λ in $\mathbb{R}^n$ such that $\|(I - E_x(\Lambda))F\psi\| < \frac{1}{2}\varepsilon$. Also since ϕ is a scattering state we have $\|E_x(\Lambda)\phi_t\| < \frac{1}{2}\varepsilon$ for all sufficiently large t.

Since V_t commutes with $E_x(\Delta)$ it follows that for all sufficiently large t

$$\begin{aligned}|\langle\phi\,|\,U_t^{0\dagger}FU_t^0\psi\rangle| &= |\langle U_t^0\phi\,|\,V_tF\psi\rangle|\\ &\leq |\langle E_x(\Lambda)U_t^0\phi\,|\,V_tF\psi\rangle| + |\langle U_t^0\phi\,|\,(I - E_x(\Lambda))V_tF\psi\rangle|\\ &\leq \|E_x(\Lambda)U_t^0\phi\| + \|(I - E_x(\Lambda))F\psi\|\\ &< \varepsilon\end{aligned}$$

and the proof is complete.

Theorem 3. $\mathscr{A}^w$ is closed under the adjoint operation but is not an operator algebra.

Proof.

(i) Obviously $A \in \mathscr{A}^w \Rightarrow A^{\dagger} \in \mathscr{A}^w$.

(ii) For any Borel set Λ in $\mathbb{R}^n$ let $\hbar\Delta = \{\hbar\boldsymbol{x}\colon \boldsymbol{x} \in \Lambda\}$; then $E_x(\Lambda) = FE_p(\hbar\Lambda)F^{\dagger}$ (Weidman 1980), F being the Fourier–Plancherel operator. So if $\mathscr{A}^w$ were an algebra it would contain each $E_x(\Lambda)$ since these are products of elements of $\mathscr{A}^w$ contradicting theorem 1.

Perhaps we should mention here that the units of various quantities are assumed fixed at the outset so that their values, e.g. $\hbar$, are specified by dimensionless numbers.

Theorem 4. $\mathscr{A}_0^w$ is closed under the adjoint operation but is not an operator algebra.

Proof.

(i) Obviously $A \in \mathscr{A}_0^w \Rightarrow A^{\dagger} \in \mathscr{A}_0^w$.

(ii) We shall show that there is a self-adjoint operator A in $\mathscr{A}_0^w$ whose square A^2 is not in $\mathscr{A}_0^w$.

From the fact that $\lim\|E_x(\Lambda)\phi_t\| = 0$ for all $\phi \in \mathscr{H}$ and all bounded Borel sets Λ it follows that

$$A = F^{\dagger}E_x(\Lambda) + E_x(\Lambda)F \in \mathscr{A}_0^w.$$

But since

$$|\langle\phi_t|F^{\dagger}E_x(\Lambda)F^{\dagger}E_x(\Lambda)\phi_t\rangle| \leq \|\phi\|\|E_x(\Lambda)\phi_t\|,$$

$$|\langle\phi_t\,|\,E_x(\Lambda)FE_x(\Lambda)F\phi_t\rangle| \leq \|\phi\|\|E_x(\Lambda)\phi_t\|,$$

and U_t^0 commutes with $E_p(\hbar\Lambda) = F^{\dagger}E_x(\Lambda)F$ it is easily verified that

$$\lim\langle\phi_t\,|\,A^2\phi_t\rangle = \lim\langle\phi_t\,|\,F^{\dagger}E_x(\Lambda)F\phi_t\rangle = \langle\phi\,|\,F^{\dagger}E_x(\Lambda)F\phi\rangle$$

which is clearly non-zero for some ϕ so $A^2 \notin \mathscr{A}_0^w$.

3. Some asymptotic algebras

We shall introduce two types of asymptotic algebra here, the first being a 'local' algebra, the second containing 'non-local' operators.

3.1. Algebras of local operators

Local observables associated with a quantum system having infinitely many degrees of freedom have been the subject of intensive studies (Emch 1972a, Bratteli and Robinson 1979, 1981) ever since Haag and Kastler (1964) introduced the algebraic approach to quantum field theory in terms of local observables. There have also been attempts to apply this algebraic approach to tackle quantum mechanical problems (Hepp 1972, Emch 1972b, Emch and Whitten-Wolfe 1976).

The specific problems on the existence of local observables and on the localisation of observables in quantum mechanics have recently been studied by Wan and his coworkers (Wan and McFarlane 1981, Wan and Jackson 1983, Wan *et al* 1983). Here we shall list some relevant mathematical properties.

Definition 3. Algebras of local operators.

(1) The local algebra in Λ is defined to be

$$\mathscr{A}_\Lambda = \{E_x(\Lambda)AE_x(\Lambda) \colon A \in B(\varkappa)\}$$

and its self-adjoint elements are called local observables in Λ.

(2) The local algebra is defined to be

$$\mathscr{A}_L = \bigcup_\Lambda \mathscr{A}_\Lambda$$

the union being taken over all bounded Borel sets Λ. The self-adjoint elements of $\mathscr{A}_L$ are called local observables.

(3) The quasi-local algebra is the uniform closure $\bar{\mathscr{A}}_L$ of $\mathscr{A}_L$ and its self-adjoint elements are called quasi-local observables.

Some mathematical properties of these algebras are given in the following theorem, while physical properties of local operators are given by Wan and Jackson (1983).

Theorem 5.

(1) If Λ is bounded then $\mathscr{A}_\Lambda$ is a proper C^*-subalgebra of $B(\mathscr{H})$.

(2) $\mathscr{A}_L$ is a proper $*$-subalgebra of $B(\mathscr{H})$ but is not a C^*-subalgebra.

(3) The strong closure of $\mathscr{A}_L$ is $B(\mathscr{H})$ and the von Neumann algebra generated by $\mathscr{A}_L$ is $B(\mathscr{H})$.

(4) Let $A \in B(\mathscr{H})$; then these statements are equivalent:

(a) $A \in \bar{\mathscr{A}}_L$.

(b) There is a sequence Λ_l of bounded Borel sets in $\mathbb{R}^n$ such that $E_x(\Lambda_l)AE_x(\Lambda_l)$ converges uniformly to A.

(c) The sequence $E_x(b_l)AE_x(b_l)$ converges uniformly to A for every increasing sequence b_l of balls converging to $\mathbb{R}^n$.

(5) $\bar{\mathscr{A}}_L$ is a proper C^*-subalgebra of $B(\mathscr{H})$ without a unit.

(6) $\bar{\mathscr{A}}_L$ contains all compact operators and in particular all finite dimensional projectors.

Proof.

(1) This is easily verified as is the assertion that $\mathscr{A}_L$ is a proper $*$-subalgebra.

(3) Let $A \in B(\mathscr{H})$ and let Λ_l be an increasing sequence of bounded Borel sets converging to $\mathbb{R}^n$. Since $E_x(\Lambda_l)$ converges strongly to I we have, using the multiplicative property of strongly convergent sequences, that $E_x(\Lambda_l)AE_x(\Lambda_l)$ converges strongly to A.

This together with the fact that a von Neumann algebra is closed in the strong operator topology establishes (3).

(4) The following inequality will be used frequently:

Let δ_1, δ_2 be two Borel sets such that $\delta_1 \subset \delta_2$; then for any $A, B \in (\mathscr{H})$ we have

$$\begin{aligned}\|E_x(\delta_2)AE_x(\delta_2)-A\| &\leq \|E_x(\delta_2)AE_x(\delta_2)-E_x(\delta_2)E_x(\delta_1)BE_x(\delta_1)E_x(\delta_2)\| \\ &\quad + \|E_x(\delta_2)E_x(\delta_1)BE_x(\delta_1)E_x(\delta_2)-A\| \\ &\leq 2\|E_x(\delta_1)BE_x(\delta_1)-A\|.\end{aligned}$$

Firstly we show (a)$\Rightarrow$(b).

Let $A \in \bar{\mathscr{A}}_L$; then there is a sequence A_l of local operators converging uniformly to A. Clearly for each l there is a bounded Borel set Λ_l with $A_l = E_x(\Lambda_l)A_lE_x(\Lambda_l)$.

Now take $\delta_1 = \delta_2 = \Lambda$ and $B = A_l$ in the above inequality giving

$$\begin{aligned}\|E_x(\Lambda_l)AE_x(\Lambda_l)-A\| &\leq 2\|E_x(\Lambda_l)A_lE_x(\Lambda_l)-A\| \\ &= 2\|A_l - A\|,\end{aligned}$$

and (b) follows.

Next we establish (b)$\Rightarrow$(c).

Assume (b) holds and define $\Delta_l = \bigcup_{k=1}^{l} \Lambda_l$; then Δ_l is an increasing sequence of bounded Borel sets and $\Lambda_l \subset \Delta_l$ for each l.

Now let b_k be an increasing sequence of balls converging to $\mathbb{R}^n$. Then we can find a strictly increasing subsequence b_{k_l} $(l = 1, 2, 3 \ldots)$ such that for each l, $\Delta_l \subset b_{k_l}$.

Now since $\Lambda_l \subset \Delta_l \subset b_{k_l}$ taking $A = B$ in the inequality obtained at the beginning gives

$$\begin{aligned}\|E_x(b_{k_l})AE_x(b_{k_l})-A\| &\leq 2\|E_x(\Lambda_l)AE_x(\Lambda_l)-A\| \\ &\to 0 \quad (\text{as } l \to \infty).\end{aligned}$$

Similarly for $k_l \leq k < k_{l+1}$ we have

$$\begin{aligned}\|E_x(b_k)AE_x(b_k)-A\| &\leq 2\|E_x(b_{k_l})AE_x(b_{k_l})-A\| \\ &\to 0 \quad (\text{as } l \to \infty).\end{aligned}$$

Hence (c) follows. The proof of (4) is complete on observing that (c)$\Rightarrow$(a) is obvious.

(5) Since $\mathscr{A}_L$ is a $*$-subalgebra its norm closure $\bar{\mathscr{A}}_L$ is a C^*-subalgebra. Suppose $I \in \bar{\mathscr{A}}_L$,; then by (4) there is a sequence Λ_l of bounded Borel sets with $\|E_x(\Lambda_l)IE_x(\Lambda_l)-I\| \to 0$ as $l \to \infty$. But

$$\|E_x(\Lambda_l)IE_x(\Lambda_l)-I\| = \|E_x(\Lambda_l)-I\| = \|E_x(\mathbb{R}^n - \Lambda_l)\| = 1$$

giving a contradiction so $I \notin \bar{\mathscr{A}}_L$.

It follows from (3) that the only operators commuting with each element of $\mathscr{A}_L$ are multiples of I. Hence $\bar{\mathscr{A}}_L$ does not have a unit.

(6) Let Λ_l be an increasing sequence of bounded Borel sets converging to $\mathbb{R}^n$. Let $\phi \in \mathscr{H}$ be a unit vector and let $\phi_l = E_x(\Lambda_l)\phi$. Then the operator $P_l = |\phi_l\rangle\langle\phi_l|$ is obviously in $\mathscr{A}_L$. Since ϕ_l converges to ϕ in the norm it follows that P_l converges uniformly to

the projector $P=|\phi\rangle\langle\phi|$, for

$$\begin{aligned}\|P_l-P\| &= \sup\|\langle\phi_l\,|\,\psi\rangle\phi_l-\langle\phi\,|\,\psi\rangle\phi\| \\ &\leq \sup(\|\langle\phi_l-\phi|\psi\rangle\phi_l\|+\|\langle\phi\,|\,\psi\rangle(\phi_l-\phi)\|) \\ &\leq \|\phi_l-\phi\|\|\phi_l\|+\|\phi_l-\phi\| \\ &\to 0 \quad \text{as } l\to\infty\end{aligned}$$

where the supremum is over all unit vectors $\psi\in\mathscr{H}$. Hence P is in $\bar{\mathscr{A}}_L$.

It follows that $\bar{\mathscr{A}}_L$ contains all one-dimensional projectors and hence all finite-dimensional projectors. Now it follows from the spectral theorem for compact self-adjoint operators (Weidmann 1980) that $\bar{\mathscr{A}}_L$ contains all compact self-adjoint operators. Since an arbitrary compact operator is of the form $A+\mathrm{i}B$ with A and B compact and self-adjoint (6) follows.

(2) Let $\psi(\boldsymbol{x})=\exp(-\boldsymbol{x}^2)$ $(\boldsymbol{x}\in\mathbb{R}^n)$ and let P be the projector associated with ψ. By (6) we have $P\in\bar{\mathscr{A}}_L$ and it is easily verified that $P\notin\mathscr{A}_L$.

3.2. Algebras containing non-local operators

Definition 4. Non-local algebras. Let u, s and w denote the uniform, strong and weak topologies on $B(\mathscr{H})$. When τ is one of these topologies we write $\tau^*\text{-lim}\, A_t=A$ and say $\tau^*\text{-lim}\, A_t$ exists whenever $\tau\text{-lim}\, A_t$ and $\tau\text{-lim}\, A_t^\dagger$ both exist and have the values A and $A^\dagger$ respectively. Note that u^* and w^* convergence coincide with uniform and weak convergence. We define

$$\mathscr{A}^\tau=\{A\in B(\mathscr{H})\colon \tau^*\text{-lim}\, A_t \text{ exists}\},$$

$$\mathscr{A}_0^\tau=\{A\in B(\mathscr{H})\colon \tau^*\text{-lim}\, A_t=0\},$$

$$\mathscr{A}_{\text{avc}}^\tau=\{A\in B(\mathscr{H})\colon \tau^*\text{-lim}\, E_{\boldsymbol{p}}(\Lambda_1)A_tE_{\boldsymbol{p}}(\Lambda_2)=0 \text{ for all disjoint Borel sets } \Lambda_1 \text{ and } \Lambda_2\}.$$

The subscript avc stands for asymptotically vanishing correlations. The reason for this terminology is the easily verified result that $\lim\langle\phi_t|A\psi_t\rangle=0$ for every $A\in\mathscr{A}_{\text{avc}}^\tau$ whenever ϕ and ψ are asymptotically separable states (as defined in Wan and McLean 1983a, b). In other words an operator in $\mathscr{A}_{\text{avc}}^\tau$ does not correlate states which are infinitely apart in space.

The significance of operators of asymptotically vanishing correlations will emerge in an alternative formulation of quantum mechanics to be presented in an ensuing paper (Wan and McLean 1984).

The following notation will be useful in stating our main theorems. Let $L^\infty(\mathbb{R}^n)$ be the set of all complex valued functions on $\mathbb{R}^n$ which are essentially bounded and measurable with respect to Lebesgue measure (Kato 1966, Roman 1975, Weidmann 1980). We denote by $L^\infty(\boldsymbol{x})$ and $L^\infty(\boldsymbol{p})$ the sets of operators of the form $f(\boldsymbol{x})$ and $f(\boldsymbol{p})$ respectively where $f\in L^\infty(\mathbb{R}^n)$. Similarly $L^\infty(H_0)$ will denote the set of all operators of the form $f(H_0)$ with $f\in L^\infty([0,\infty))$.

Let us now study the families of operators introduced above in each topology. Beginning with the weak topology we firstly recover $\mathscr{A}^w$ and $\mathscr{A}_0^w$ already discussed in the previous section. The only new element is the set $\mathscr{A}_{\text{avc}}^w$ whose properties are given by the lemmas and theorem below.

Lemma 2. The parity operator $\mathscr{P}$ defined by $\mathscr{P}\phi(\boldsymbol{x})=\phi(-\boldsymbol{x})$ $(\forall\phi\in\mathscr{H})$ is not a member of $\mathscr{A}_0^w\cup\mathscr{A}_{\text{avc}}^w$.

Proof. It is well known that $[\mathscr{P}, H_0]=0$ (Roman 1965). We have therefore $\langle\phi_t|\mathscr{P}\psi_t\rangle = \langle\phi|\mathscr{P}\psi\rangle$ which obviously does not vanish for some ϕ, ψ, or for some ϕ, ψ with $E_p(\Lambda_1)\phi=\phi, E_p(\Lambda_2)\psi=\psi$ where Λ_1 and Λ_2 are disjoint. We conclude that $\mathscr{P}\notin\mathscr{A}_0^w\cup\mathscr{A}_{\text{avc}}^w$.

Theorem 6.

(1) $\mathscr{A}_{\text{avc}}^w\supset\mathscr{A}_0^w\supset\bar{\mathscr{A}}_L$.

(2) $\mathscr{A}_{\text{avc}}^w$ is not an operator algebra.

(3) $\mathscr{A}_{\text{avc}}^w$ does not contain $\mathscr{A}^w$.

(4) $\mathscr{A}_{\text{avc}}^w\cap\mathscr{A}^w=\mathscr{A}_0^w+L^\infty(\boldsymbol{p})$.

(5) $\mathscr{A}_{\text{avc}}^w\cap\mathscr{A}^w$ is not an operator algebra.

Proof.

(1) $\mathscr{A}_0^w$ is obviously in $\mathscr{A}_{\text{avc}}^w$. Now let $A\in\bar{\mathscr{A}}_L$; then there exists a sequence A_l in $\mathscr{A}_L$ converging uniformly to A. We have $\lim_{t\to\infty}\langle\phi_t|A\phi_t\rangle=\lim_{t\to\infty}\lim_{l\to\infty}\langle\phi_t|A_l\phi_t\rangle=0$.

(2) The Fourier–Plancherel operator F is in $\mathscr{A}_{\text{avc}}^w$ since F is in $\mathscr{A}_0^w$ by lemma 1. If $\mathscr{A}_{\text{avc}}^w$ were an operator algebra, F^2 would be in $\mathscr{A}_{\text{avc}}^w$. But $F^2=\mathscr{P}$, the parity operator (Weidmann 1980) which is not in $\mathscr{A}_{\text{avc}}^w$ by lemma 2. Hence we establish (2).

(3) The parity operator is in $\mathscr{A}^w$ but not in $\mathscr{A}_{\text{avc}}^w$.

(4) Every operator in $\mathscr{A}_0^w+L^\infty(\boldsymbol{p})$ is obviously in $\mathscr{A}_{\text{avc}}^w\cap\mathscr{A}^w$. Now let $A\in\Delta$ $\mathscr{A}_{\text{avc}}^w\cap\mathscr{A}^w$ and let $B=w\text{-lim}\,A_t$. Then $w\text{-lim}(E_p(\Lambda)AE_p(\Lambda^c))_t=E_p(\Lambda)BE_p(\Lambda^c)=0$, where Λ^c is $\mathbb{R}^n-\Lambda$. Consequently we have

$$BE_p(\Lambda)=(E_p(\Lambda)+E_p(\Lambda^c))BE_p(\Lambda)=E_p(\Lambda)BE_p(\Lambda),$$

$$E_p(\Lambda)B=E_p(\Lambda)B(E_p(\Lambda)+E_p(\Lambda^c))=E_p(\Lambda)BE_p(\Lambda).$$

Hence $[B, E_p(\Lambda)]=0$. It follows that B belongs to $L^\infty(\boldsymbol{p})$ (Dixmier 1969) and so (4) in the theorem is established.

(5) To prove statement (5) we can argue as in (2).

Let us now proceed to examine the situation in the uniform topology. Let $\{H_0\}'$ denote the set of all bounded operators which commute with H_0; then we have the following theorem.

Theorem 7.

(1) $\mathscr{A}_0^u=\{0\}$, $\mathscr{A}^u=\{H_0\}'$, $\mathscr{A}_{\text{avc}}^u=L^\infty(\boldsymbol{p})$.

(2) $\mathscr{A}^u$ and $\mathscr{A}_{\text{avc}}^u$ are von Neumann algebras on $\mathscr{H}$ and

$$\mathscr{A}_0^u\subset\mathscr{A}_{\text{avc}}^u\subset\mathscr{A}^u\subset B(\mathscr{H}).$$

Proof.

(1)(i) $A\in\mathscr{A}_0^u\Leftrightarrow\lim\|U_t^{0+}AU_t^0\|=0\Leftrightarrow\|A\|=0$.

(ii) $\{H_0\}'\subset\mathscr{A}^u$ is easily proved.

Conversely suppose $A\in\mathscr{A}^u$ and let $L=u\text{-lim}\,A_t$; then $\forall s\in\mathbb{R}$

$$\begin{aligned}\|L-L_s\|&\leq\|L-A_{s+t}\|+\|A_{s+t}-L_s\| \qquad (L_s=U_s^{0+}LU_s^0)\\&=\|L-A_{s+t}\|+\|A_t-L\|,\end{aligned}$$

so

$$\|L-L_s\|\leq\lim_{t\to\infty}(\|L-A_{s+t}\|+\|A_t-L\|)=0.$$

Hence $(\forall s\in\mathbb{R})$ $U_s^0L=LU_s^0$ and it follows from Stone's theorem that L commutes with H_0.

Finally if E is any spectral projector of H_0 then L commutes with E and E commutes with U_t^0 so

$$\begin{aligned}\|AE-EA\| &= \lim\|U_t^{0\dagger}(AE-EA)U_t^0\| = \lim\|A_tE-EA_t\| \\ &= \|LE-EL\| = 0\end{aligned}$$

and it follows that A commutes with H_0.

(iii) Making use of the property $E_p(\Lambda)U_t^0 = U_t^0E_p(\Lambda)$ we have

$$\begin{aligned} A\in L^\infty(\boldsymbol{p}) &\Rightarrow \forall \text{ disjoint } \Lambda_1,\Lambda_2\ \|E_p(\Lambda_1)A_tE_p(\Lambda_2)\| = \|E_p(\Lambda_1)E_p(\Lambda_2)A\| = 0 \\ &\Rightarrow A\in\mathscr{A}^u_{\text{avc}}. \\ A\in\mathscr{A}^u_{\text{avc}} &\Rightarrow (\forall\Lambda)\ \lim\|(I-E_p(\Lambda))A_tE_p(\Lambda)\| = 0 \\ &\Rightarrow (\forall\Lambda)\ \|(I-E_p(\Lambda))AE_p(\Lambda)\| = 0 \\ &\Rightarrow (\forall\Lambda)\ AE_p(\Lambda) = E_p(\Lambda)AE_p(\Lambda) = E_p(\Lambda)A \\ &\Rightarrow A \text{ commutes with every } E_p(\Lambda) \\ &\Rightarrow A\in L^\infty(\boldsymbol{p}) \text{ (Dixmier 1969).}\end{aligned}$$

(2) That each set is a von Neumann algebra follows from (1) and the inclusion $\mathscr{A}_0^u\subset\mathscr{A}^u_{\text{avc}}$ is obvious. It is easily checked that the parity operator commutes with H_0 but not with $\boldsymbol{p}$ so $\mathscr{A}^u_{\text{avc}}\subset\mathscr{A}^u$.

Finally observe that $\mathscr{A}^u$ contains $L^\infty(H_0)$ as a proper subset, e.g. the parity operator is in $\mathscr{A}^u$ but not in $L^\infty(H_0)$. Note also that $A_t=A$ for every A in $\mathscr{A}^u$. In other words A in $\mathscr{A}^u$ does not evolve in time.

We shall now turn our attention to the situation in the strong topology. The results obtained are summarised in the following theorem which is of immediate physical interest (Wan and McLean 1984).

Theorem 8. Let $\mathscr{A}=\mathscr{A}^s\cap\mathscr{A}^s_{\text{avc}}$; then

(1) $\mathscr{A}^s$ is a proper C^*-subalgebra of $B(\mathscr{H})$ containing the parity operator.

(2) $\mathscr{A}=\{A\in B(\mathscr{H})\colon s^*\text{-lim } A_t \text{ exists and belongs to } L^\infty(\boldsymbol{p})\}=\mathscr{A}_0^s+L^\infty(\boldsymbol{p})$ and both $\mathscr{A}_0^s$ and $\mathscr{A}$ are C^*-subalgebras of $B(\mathscr{H})$.

(3) $\mathscr{A}^s$ is an asymptotic algebra and

$$\bar{\mathscr{A}}_L\subset\mathscr{A}_0^s\subset\mathscr{A}\subset\mathscr{A}^s\subset B(\mathscr{H}).$$

Proof.

(1) Since the parity operator is self-adjoint and commutes with U_t^0 it clearly belongs to $\mathscr{A}^s$. $\mathscr{A}^s$ is a proper subset of $B(\mathscr{H})$ by theorem 1.

The only remaining parts of the assertion which are not obvious are that $\mathscr{A}^s$ is closed under multiplication and is closed in the uniform topology.

If $A, B\in\mathscr{A}^s$ with $T=s\text{-lim } A_t$ and $S=s\text{-lim } B_t$ then the inequality

$$\begin{aligned}\|((AB)_t-TS)\phi\| &\le \|(A_tB_t-A_tS)\phi\|+\|(A_tS-TS)\phi\| \\ &\le \|A\|\|(B_t-S)\phi\|+\|(A_t-T)S\phi\|\end{aligned}$$

implies that $s\text{-}\lim(AB)_t = TS$. Since A and B are arbitrary and $T^\dagger = s\text{-}\lim A_t^\dagger$, $S^\dagger = s\text{-}\lim B_t^\dagger$ we have $s\text{-}\lim(AB)_t^\dagger = s\text{-}\lim(B_t^\dagger A_t^\dagger) = S^\dagger T^\dagger$ and it follows that $AB \in \mathscr{A}^s$.

Let A_j be a sequence in $\mathscr{A}^s$ converging uniformly to $A \in B(\mathscr{H})$ and for each j let $T_j = s\text{-}\lim_{t\to\infty}(A_j)_t$; then for any $\phi \in \mathscr{H}$

$$\|(T_j - T_k)\phi\| = \lim_{t\to\infty}\|(A_j - A_k)\phi_t\|$$

$$\leqslant \|\phi\|\|A_j - A_k\|.$$

Hence T_j is strongly Cauchy so $T = s\text{-}\lim_{j\to\infty} T_j$ exists (Weidmann 1980).

Now fix $\phi \in \mathscr{H}$ and define

$$f_j(t) = \|(A_j)_t\phi - T_j\phi\|, \qquad f(t) = \|A_t\phi - T\phi\|;$$

then

$$\sup_{t\in\mathbb{R}}|f_j(t) - f(t)| \leqslant \|A_j - A\|\|\phi\| + \|(T - T_j)\phi\|,$$

so f_j converges uniformly to f and we have

$$\lim_{t\to\infty}\|(A_t - T)\phi\| = \lim_{j\to\infty}\lim_{t\to\infty} f_j(t) = 0.$$

Thus $s\text{-}\lim A_t$ exists. Since $A_j^\dagger$ converges uniformly to $A^\dagger$ an identical argument gives the existence of $s\text{-}\lim A_t^\dagger$, so $A \in \mathscr{A}^s$ as required.

(2) Let $\mathscr{M}$ denote the set on the right-hand side of the first equality in (2) of the theorem. We show $\mathscr{A} \subset \mathscr{M} \subset \mathscr{A}_0^s + L^\infty(\boldsymbol{p}) \subset \mathscr{A}$.

Firstly let $A \in \mathscr{A}$ and let $T = s\text{-}\lim A_t$, then since $A \in \mathscr{A}_{\text{avc}}^s$ we have for any Borel set Λ

$$E_{\boldsymbol{p}}(\Lambda)TE_{\boldsymbol{p}}(\Lambda^c) = s\text{-}\lim E_{\boldsymbol{p}}(\Lambda)A_tE_{\boldsymbol{p}}(\Lambda^c) = 0$$

where $\Lambda^c = \mathbb{R}^n - \Lambda$, and similarly $E_{\boldsymbol{p}}(\Lambda^c)TE_{\boldsymbol{p}}(\Lambda) = 0$. Now for any Λ

$$\begin{aligned} TE_{\boldsymbol{p}}(\Lambda) &= (E_{\boldsymbol{p}}(\Lambda) + E_{\boldsymbol{p}}(\Lambda^c))TE_{\boldsymbol{p}}(\Lambda) \\ &= E_{\boldsymbol{p}}(\Lambda)TE_{\boldsymbol{p}}(\Lambda) \\ &= E_{\boldsymbol{p}}(\Lambda)T(E_{\boldsymbol{p}}(\Lambda) + E_{\boldsymbol{p}}(\Lambda^c)) \\ &= E_{\boldsymbol{p}}(\Lambda)T, \end{aligned}$$

and it follows that $T \in L^\infty(\boldsymbol{p})$ (Dixmier 1969). Similarly $s\text{-}\lim A_t^\dagger$ belongs to $L^\infty(\boldsymbol{p})$, so $A \in \mathscr{M}$.

Let $A \in \mathscr{M}$ and let $T = s\text{-}\lim A_t$. Then $T \in L^\infty(\boldsymbol{p})$ so T commutes with each U_t^0 and we have

$$\|(A - T)_t\phi\| = \|(A_t - T)\phi\| \to 0 \qquad \forall \phi \in \mathscr{H}.$$

Also since $s\text{-}\lim A_t^\dagger$ exists it must equal $T^\dagger$ so similarly

$$\|(A^\dagger - T^\dagger)_t\phi\| \to 0.$$

Hence $A - T \in \mathscr{A}_0^s$ which implies $A \in \mathscr{A}_0^s + L^\infty(\boldsymbol{p})$.

Finally suppose $A \in \mathscr{A}_0^s \dot{+} L^\infty(\boldsymbol{p})$; then $A = T + G$ for some $T \in \mathscr{A}_0^s$ and some $G \in \Delta$ $L^\infty(\boldsymbol{p})$. It is easily checked that $T \in \mathscr{A}^s \cap \mathscr{A}_{\text{avc}}^s$ and since G commutes with each U_t^0 and $E_{\boldsymbol{p}}(\Lambda)$ we have $G \in \mathscr{A}^s \cap \mathscr{A}_{\text{avc}}^s$ and it follows that $A \in \mathscr{A}$.

To complete the proof of (2) we need to show that $\mathscr{A}_0^s$ and $\mathscr{A}$ are C^*-subalgebras of $B(\mathscr{H})$. The only properties which are not obvious are that each is closed under multiplication and closed in the uniform topology.

Let A_j be a sequence in $\mathscr{A}_{\text{avc}}^s$ converging uniformly to A. Let Λ_1, Λ_2 be disjoint Borel sets and let $\phi \in \mathscr{H}$; then the following swopping of limits is easily justified by uniform convergence

$$\lim_{t\to\infty}\|E_p(\Lambda_1)A_tE_p(\Lambda_2)\| = \lim_{j\to\infty}\lim_{t\to\infty}\|E_p(\Lambda_1)(A_j)_tE_p(\Lambda_2)\| = 0.$$

Thus $A \in \mathscr{A}_{\text{avc}}^s$, so $\mathscr{A}_{\text{avc}}^s$ is closed and hence $\mathscr{A} = \mathscr{A}^s \cap \mathscr{A}_{\text{avc}}^s$ is closed (using (1)).

Similarly if A_j is a sequence in $\mathscr{A}_0^s$ converging uniformly to A then for every $\phi \in \mathscr{H}$

$$\lim_{t\to\infty}\|A\phi_t\| = \lim_{j\to\infty}\lim_{t\to\infty}\|A_j\phi_t\| = 0,$$

so $\mathscr{A}_0^s$ is closed.

It follows from the first inequality in the proof of (1) that $\mathscr{A}_0^s$ and $\mathscr{M}$ are closed under multiplication. This completes the proof that $\mathscr{A}_0^s$ and $\mathscr{A} = \mathscr{M}$ are C^*-subalgebras.

(3) $\mathscr{A}^s$ is an asymptotic algebra by (2) and the fact that strong convergence implies weak convergence.

Let $A \in \mathscr{A}_L$; then $A = E_x(\Lambda)AE_x(\Lambda)$ for some bounded Borel set Λ and since every $\phi \in \mathscr{H}$ is a scattering state of the free particle Hamiltonian we have

$$\|A_t\phi\| = \|E_x(\Lambda)AE_x(\Lambda)\phi_t\| \leqslant \|A\|\|E_x(\Lambda)\phi_t\| \to 0.$$

Hence $\mathscr{A}_L \subset \mathscr{A}_0^s$ and so $\bar{\mathscr{A}}_L \subset \mathscr{A}_0^s$ since $\mathscr{A}_0^s$ is closed by (2).

Again using (2) each step in the following is obvious

$$\mathscr{A}_0^s \subset \mathscr{A}_0^s + L^\infty(\boldsymbol{p}) = \mathscr{A}^s \cap \mathscr{A}_{\text{avc}}^s \subset \mathscr{A}^s \subset B(\mathscr{H}).$$

4. Concluding remarks

Quantum mechanics is troubled by the problem of non-locality (Selleri and Tarozzi 1981) inherent in the theory. We explore the possibility of an alternative formulation of quantum mechanics which would incorporate non-locality when small distances are involved but would be separable at large distances. A main purpose of studying asymptotic operator algebras is to establish a proper C^*-subalgebra of $B(\mathscr{H})$ which can be associated with a quantum mechanical system and on which an alternative algebraic formulation of quantum mechanics with the desired non-locality and separability characteristics may be established. In the light of results obtained so far we are led naturally to the algebra $\mathscr{A}_0^s + L^\infty(\boldsymbol{p})$. A detailed formulation of quantum mechanics based on this algebra is presented in an ensuing paper (Wan and McLean 1984).

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